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Fluxes in M-theory on 7-manifolds and G structures

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Abstract: We consider warp compactifications of M-theory on 7-manifolds in the presence of 4-form fluxes and investigate the constraints imposed by supersymmetry. As long as the 7-manifold supports only one Killing spinor we infer from the Killing spinor equations that non-trivial 4-form fluxes will necessarily curve the external 4-dimensional space. On the other hand, if the 7-manifold has at least two Killing spinors, there is a non-trivial Killing vector yielding a reduction of the 7-manifold to a 6-manifold and we confirm that 4-form fluxes can be incorporated if one includes non-trivial SU(3) structures.

Keywords: M-Theory, Superstring Vacua, Supersymmetric Effective Theories, Differential and Algebraic Geometry.

Contents

1. Introduction

One way to describe phenomenological interesting models in 4 dimensions with $\mathcal{N}$=1 supersymmetry is to consider M-theory on a 7-manifold with G_2 holonomy. In this case the 4-form field is trivial, but one may ask whether one can turn on a non-trivial 4-form flux in the internal space while still keeping the flat 4-d Minkowski space with four unbroken supercharges. Over the past years this question has been explored in different directions with more or less restrictive assumptions. In a number of papers the existence of BPS solutions with non-trivial fluxes are excluded, see e.g. [1]–[5], or a non-trivial superpotential appears see e.g. [6, 7]. Other no-go theorems base on discussions of the equations of motion [8]. But most of the no-theorem statements have restrictive assumptions, as e.g. a compact internal space without sources or a semi-definite potential. In many models this is not the case and in fact, in the literature one can find examples of M-theory compactifications in the presence of fluxes [9]–[13] as well as examples of 10-d string theory with fluxes that yield a flat 4-dimensional vacuum [14]–[17]. The essential ingredient of these string theory compactifications are non-trivial SU(3) structures (i.e. torsion) as well as a warped geometry, see also [18, 19]. Moreover, it is well-known that one can compactify M-theory in presence of 4-form fluxes to a $D = 4$, $\mathcal{N} = 1$ anti de Sitter vacuum [20], i.e. the corresponding superpotential has a non-vanishing extremum. Similar to the string theory compactifications, this solution involves non-trivial G structures, where the torsion 3-form, that parallelizes the 7-manifold (deformed $\mathbb{S}^7$), is the dual of the 4-form on the 7-manifold [21].

In this note we attempt to clarify different aspects. We will especially relax the assumption, made in a number of papers, that the 11-dimensional spinor is a direct product of the 4-spinor and 7-spinor, see also [20, 5]. With this in mind, we can summarize our assumptions as follows. We are looking for M-theory configurations that allow upon (warp) compactifications a flat Minkowski space with four unbroken supercharges. In order to keep Lorentz symmetry, we assume that all Kaluza-Klein vector fields are trivial yielding a block-diagonal form of the metric and allowing only for internal components of the 4-form field. Our ansatz for the 11-d metric and 4-form field strength reads therefore

$$\begin{aligned} ds^2 &= e^{2A}\,\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2B}\,h_{mn}dy^m dy^n \\ F &= F_{mnpq}\,dy^m \wedge dy^n \wedge dy^p \wedge dy^q \end{aligned} \tag{1.1}$$

where $A = A(y)$ and $B = B(y)$ are functions of the coordinates of the 7-manifold with the metric h_{mn}. We omitted Lorentz-invariant 4-form field components like $F_{\mu\nu\rho\lambda} \sim \epsilon_{\mu\nu\rho\lambda}$, which cannot be embedded into a flat Minkowski space and yield an anti de Sitter vacuum. As we will see the 11-d Killing spinor equations can be solved only, if the 7-manifold described by the metric h_{mn}: (i) has G_2-holonomy with trivial fluxes and warp factors or (ii) allows for more than one Killing spinors, which implies that the 7-manifold has to have a Killing vector and yields effectively a reduction to a 6-manifold. It is important to realize that supersymmetry requires non-trivial SU(3) structures on this 6-manifold which are related to a non-trivial antisymmetric tensor field; otherwise warp compactifications[1] of 10-dimensional string models are known to break supersymmetry or yield a non-flat 4-dimensional vacuum [2]. But before we come to the discussion of this issue, we will start with general remarks about Killing spinors and the resulting holonomy.

2. Covariantly constant spinors and holonomy

We use the convention $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ with $\eta = \mathrm{diag}(-,+,+\ldots+)$ and we decompose the 11-d Γ-matrices as (see also [1])

$$\Gamma^\mu = \hat{\gamma}^\mu \otimes \mathbb{I}\,, \qquad \Gamma^{m+3} = \hat{\gamma}^5 \otimes \gamma^m \tag{2.1}$$

with $\mu = 0,1,2,3$, $m = 1,2,\ldots,7$. Moreover, in our conventions we have

$$\hat{\gamma}^5 = i\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3\,, \qquad \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7 = -i \tag{2.2}$$

yielding

$$\frac{i}{3!}\epsilon^{mnpqabc}\gamma_{abc} = \gamma^{mnpq} \equiv \gamma^{[m}\gamma^n\gamma^p\gamma^{q]}\,. \tag{2.3}$$

In addition to the Γ-matrices we have to decompose the 11-d spinor into an anti-commuting 4-spinor and a commuting 7-spinor. The spinor on the euclidean 7-manifold has to be (pseudo) Majorana, where the identity can be chosen as charge conjugation matrix and the γ-matrices become up to an overall factor of "i" real and antisymmetric. If this 7-manifold allows for commuting spinors θ^k, one can construct differential forms in the usual way

$$\Omega^{kl}_{a_1\cdots a_n} = (i)^n\theta^k\gamma_{a_1\cdots a_n}\theta^l\,. \tag{2.4}$$

Obviously these forms are covariantly constant if the spinor is covariantly constant, but let us stress that this does not need to be the case! Since $(\gamma_{a_1\cdots a_n})^T = (-)^{\frac{n^2+n}{2}}\gamma_{a_1\cdots a_n}$ it follows

[1]From the 4- or 5-dimensional point of view it is interesting to note the close relationship of flux or warp compactifications, gauged supergravity and the attractor mechanism, which gives an explaination of the fixing of (vector) moduli in these compactifications [22, 23, 24, 25, 26, 27].

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that this form is antisymmetric in $[k,l]$ for $n = 1,2,5,6$ and symmetric for $n = 0,3,4,7$. Thus, if there is more than one covariantly constant Killing spinor, one can always construct at least one covariantly constant vector and if the Killing spinor is globally well-defined, one can always find a coordinate system so that the corresponding U(1) fiber in the metric is trivial; or in other words the 7-manifold factorizes into $\mathbb{R} \times \mathbb{X}_6$. For the case with just two Killing spinors ($k,l = 1,2$) we obtain one covariantly constant vector and $\mathbb{X}_6$ should not be factorisable. In addition there is one 2-form and three 3-forms, for which (2.4) is symmetric in (k,l). By doing Fierz re-arrangements one can show, see [28], that the 2-form lives only on $\mathbb{X}_6$ whereas the 3-forms combine into one complex 3-form on $\mathbb{X}_6$ and one 3-form extending along $\mathbb{R}$. This identifies $\mathbb{X}_6$ as a complex-3-dimensional space with SU(3) holonomy. If there are four covariantly constant spinors ($i,j = 1..4$) the situation becomes even more involved. Now, one can construct six 1-forms and six 2-forms, which are consistent with the splitting $\mathbb{R}^3 \times \mathbb{CY}_2$. In fact, regarding the 7-space as a fibration of a 3-space over a 4-space, three of the covariantly constant vectors make the fibration trivial yielding a product space and the remaining three 1-forms ensure that the 3-space is $\mathbb{R}^3$. The six 2-forms split now in three 2-forms supported by $\mathbb{R}^3$ and the remaining 2-forms identify the 4-space as a hyper Kaehler space which has SU(2) holonomy. We do not need to discuss here the case of maximal supersymmetry related to eight covariantly constant spinors, because the space becomes trivial.

Let us come back to the case of just one Killing spinor and consider first the case where this spinor is covariantly constant with respect to a metric h_{mn}, i.e.

$$0 = \nabla^{(h)}\theta \equiv \left[d + \frac{1}{4}\,\omega^{mn}\gamma_{mn}\right]\theta \tag{2.5}$$

where ω^{mn} is the spin-connection 1-form. If this equation has only one solution the holonomy group of the space must be equal to G_2 and since the spinor is covariantly constant, its holonomy is trivial and hence is a G_2 singlet. The unrestricted holonomy of an orientable 7-manifold is SO(7) and in order to decompose the adjoint representation of SO(7): $\mathbf{21} \to \mathbf{14} + \mathbf{7}$ under its maximal compact subgroup G_2 one introduces two projectors $\mathbb{P}_{\mathbf{7}/\mathbf{14}}$ [21](corresponding to the 3-form φ defined in (2.7))

$$\mathbb{P}^{pq}_{\mathbf{14}\ mn} \equiv \frac{2}{3}\left(\mathbb{I}^{pq}{}_{mn} - \frac{1}{4}\psi^{pq}{}_{mn}\right), \qquad \mathbb{P}^{pq}_{\mathbf{7}\ mn} \equiv \frac{1}{3}\left(\mathbb{I}^{pq}{}_{mn} + \frac{1}{2}\psi^{pq}{}_{mn}\right). \tag{2.6}$$

where $\mathbb{I}^{pq}{}_{mn} = \delta^p_{[m}\delta^q_{n]}$ and ψ_{mnpq} is the G_2-invariant 4-index object, which is defined in the tangent space and coincides with the covariantly constant 4-form as introduced in (2.4). This 4-form is dual to a 3-form and the requirement that both forms are closed gives equations for the vielbeine. For a given set of vielbeine e^m the 3-form can be written as

$$\begin{aligned}\varphi &= \frac{1}{3!}\varphi_{abc}\, e^a \wedge e^b \wedge e^c \\ &= e^1 \wedge e^2 \wedge e^7 + e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - \\ &\quad -e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5 + e^3 \wedge e^4 \wedge e^7 + e^5 \wedge e^6 \wedge e^7\,. \end{aligned} \tag{2.7}$$

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Both G_2-invariant forms fulfill a number of useful relations [13] and for later convenience we will note, that

$$\psi_{mnpq}\varphi^{qkl} = -6\varphi^{[k}_{\ \ [mn}\,\delta_{p]}^{\ l]}\,, \qquad \varphi_{kmn}\varphi^{mnl} = 6\delta_k^{\ l}\,. \tag{2.8}$$

The spin connection transforms as a product of a vector- and tensor-representation of SO(7), where the tensor part is just the adjoint representation. One decomposes now this tensor ($\mathbf{21} \to \mathbf{7} + \mathbf{14}$) by inserting the identity $\mathbb{I} = \mathbb{P}_{\mathbf{14}} + \mathbb{P}_{\mathbf{7}}$, i.e.

$$\left[\frac{1}{4}\omega^{mn}\,\gamma_{mn}\right]\theta = \left[\frac{1}{4}\omega^{mn}\,\mathbb{I}^{pq}{}_{mn}\,\gamma_{pq}\right]\theta$$

where $\mathbb{P}_{\mathbf{7}}$ projects onto the $\mathbf{7}$ and $\mathbb{P}_{\mathbf{14}}$ onto the $\mathbf{14}$, which is the adjoint of G_2. One solves now equation (2.5) for a constant spinor θ by requiring that the projection of the spin connection onto the $\mathbf{7}$ vanishes (so that it represents a G_2 generator) and that the spinor does not transform under G_2, i.e. the spinor is a zero mode of the G_2 generators (i.e. the $\mathbf{14}$), see also [21, 13] for more details. So, one obtains the equations

$$\mathbb{P}_{\mathbf{7}}{}^{pq}{}_{mn}\,\omega^{mn} = 0\,, \tag{2.9}$$

$$\mathbb{P}_{\mathbf{14}}{}^{pq}{}_{mn}\,\gamma_{pq}\theta = 0\,. \tag{2.10}$$

Note, the first equation gives first order differential equations for the metric h_{mn} and the second equation projects out seven of the eight spinor components. In a number of papers explicit examples have been discussed over the past years, see e.g. [29]–[32]. We will not discuss the first order differential equations for the metric, but we want to bring the projector constraint on θ in another form by multiplying it with γ^{mn} which gives

$$\gamma^{mn}\mathbb{P}_{\mathbf{14}}{}^{pq}{}_{mn}\,\gamma_{pq}\theta = \left[\mathbb{I} + \frac{1}{7\,4!}\psi^{mnpq}\gamma_{mnpq}\right]\theta = \left[\mathbb{I} + i\,\frac{1}{7\,3!}\,\varphi_{mnp}\gamma^{mnp}\right]\theta = 0 \tag{2.11}$$

where we used the relation (2.3) and $\psi = {}^{\star}\varphi$.

Up to now we were assuming that the spinors are covariantly constant with respect to the Levi-Civita connection, but this is highly restrictive. In general the 7-spinor does not need to be covariantly constant and neither are the differential forms in (2.4). This is the case if one takes into account non-trivial G_2-structures, where the deviation from the covariantly constance can be absorbed into non-trivial torsion terms entering a generalized connection. Let us summarize some basic features, for more details see e.g. [33, 34, 18, 35]. The 3- and 4-form should still be G_2 invariant and coincide in the tangent space with the expressions that we discussed so far. Also the decomposition of the SO(7) tensor representation under G_2 in terms of the projectors $\mathbb{P}_{\mathbf{7}/\mathbf{14}}$ is unchanged so that one gets the same projector acting on the spinor θ in (2.10), that again projects out seven of the eight spinor components. The inclusion of torsion means however, that the projection of the Levi-Civita connection onto the $\mathbf{7}$ is now non-vanishing and given by the different torsion classes. This means that the equation (2.9) does not vanish anymore. The torsion is given by a 3-form $H_m{}^{ab}$ which under G_2 decomposes into a $\mathbf{7}$ of the antisymmetric indices $[ab]$ as well as for the vector index m. One gets in total four torsion classes related to the decomposition: $\mathbf{7}\otimes\mathbf{7}=\mathbf{1}+\mathbf{7}+\mathbf{14}+\mathbf{27}$. In string- and M-theory the presence of RR-fluxes yields typically non-trivial G-structures for the forms defined in (2.4), see [28, 18], and we will comment more on it in the last section.

3. Solving the killing spinor equation

Unbroken supersymmetry is equivalent to the existence of (at least) one Killing spinor η yielding a vanishing gravitino variation of 11-dimensional supergravity

$$0=\delta\Psi_M=\Big[\partial_M+\frac{1}{4}\hat{\omega}_M^{RS}\Gamma_{RS}+\frac{1}{144}\Big(\Gamma_M^{\ NPQR}-8\,\delta_M^N\,\Gamma^{PQR}\Big)\,F_{NPQR}\Big]\,\eta\,. \tag{3.1}$$

Later on, we will decompose the 11-d Killing spinor into a 4- and 7-spinor and because they parameterize supersymmetry transformations we also call them Killing spinors. With our ansatz from eq. (1.1) and the conventions introduced in the last section we write the gravitino variation covariantly in the metric h_{mn} and obtain two equations

$$0=\delta\Psi_\mu=\partial_\mu\eta+\frac{1}{2}e^{A+B}\left[\hat{\gamma}_\mu\hat{\gamma}^5\otimes\gamma^m\partial_m A+\frac{1}{72}e^{3B}\hat{\gamma}_\mu\otimes F\right]\eta\,, \tag{3.2}$$

$$0=\delta\Psi_m=\nabla_m^{(h)}\eta+\frac{1}{2}\left[-\mathbb{I}\otimes\gamma_m^{\ n}\partial_n B+\frac{1}{72}e^{3B}\,\left(\hat{\gamma}^5\otimes\gamma_m F-12\hat{\gamma}^5\otimes F_m\right)\right]\eta \tag{3.3}$$

where we introduced the abbreviations

$$F\equiv F_{mnpq}\gamma^{mnpq}\,,\qquad F_m\equiv F_{mnpq}\gamma^{npq}\,. \tag{3.4}$$

and used

$$\gamma_m^{\ npqr}F_{npqr}=\gamma_m F-4F_m\,. \tag{3.5}$$

This relation is a consequence of the general formula for products of γ-matrices

$$\gamma^a\gamma_{b_1\cdots b_n}=n\,\delta^a_{[b_1}\gamma_{b_2\cdots b_n]}+\gamma^a_{\ b_1\cdots b_n}\,. \tag{3.6}$$

We should put a warning at this point. To make the notation as simple as possible, we will not make a clear distinction between curved and flat indices. Of course the γ-matrix algebra and the spinor projection is defined in the tangent space, but the 4-form field as well as derivatives are always with respect to curved indices. Having this in mind we will avoid any underlined indices and hope that it is clear from the context.

We will start with equation (3.2), which yields as integrability condition

$$\begin{aligned}0&=\left[\partial_m A\partial^m A\,(\mathbb{I}\otimes\mathbb{I})-\frac{1}{9}e^{6B}\,\left(\mathbb{I}\otimes F^2\right)+\frac{1}{9}e^{3B}\partial_m A\left(\hat{\gamma}^5\otimes F^m\right)\right]\eta\\&=\left[\mathbb{I}\otimes\gamma^{pq}\partial_q A-\frac{1}{12}e^{3B}\hat{\gamma}^5\otimes F^p\right]\gamma_p\gamma_m\left[\mathbb{I}\otimes\gamma^{mn}\partial_n A+\frac{1}{12}e^{3B}\hat{\gamma}^5\otimes F^m\right]\eta\,.\end{aligned} \tag{3.7}$$

This equation is solved if

$$\left[\mathbb{I}\otimes\gamma^{mn}\partial_n A+\frac{1}{12}e^{3B}\hat{\gamma}^5\otimes F^m\right]\eta=0\,. \tag{3.8}$$

Multiplying this equation with γ^m and inserting it back into (3.2) yields $\partial_\mu \eta = 0$, which is consistent, since we are interested in a flat Minkowski vacuum. Following [5] we will use this equation to replace the terms containing the 4-form field in equation (3.3) and find

$$\mathbb{I} \otimes \left(\nabla_m^{(h)} - \frac{1}{2} \partial_m A \right) , \eta - \frac{1}{2} \partial_n (A + B)(\mathbb{I} \otimes \gamma_m{}^n) \eta = 0 \,. \tag{3.9}$$

To solve this equation, we take now the freedom to choose the warp factor B appropriately and rescale the spinor as

$$A = -B \,, \qquad \eta = e^{\frac{A}{2}} \hat{\eta} \tag{3.10}$$

yielding for $\hat{\eta}$

$$\left(\mathbb{I} \otimes \nabla_m^{(h)} \right) \hat{\eta} = 0 \,. \tag{3.11}$$

So we have two equations, (3.8) and (3.11), that have to be satisfied simultaneously. Note, in both equations $\hat{\eta}$ is still a spinor of 11-d supergravity and we have now to decompose it into a 4-spinor ϵ and a 7-spinor θ. Note also, equation (3.11) *does not* mean that the 7-spinor is covariantly constant! Only if we choose $\hat{\eta} = \epsilon \otimes \xi$, which is often used in the literature, we could infer on a covariantly constant spinor θ, but this choice seems to be consistent only for trivial fluxes and warp factors. For non-vanishing fluxes and non-trivial warping, equation (3.8) would yield for $\hat{\eta} = \epsilon \otimes \xi$ a Weyl constraint on ϵ ($\hat{\gamma}^5 \epsilon \sim \epsilon$), but the 4-spinor should be a Majorana since the 11- as well as the 7-spinor are both Majorana spinors. Therefore, we do not consider the direct product ansatz, but instead decompose $\hat{\eta}$ in a more general way (see also [20, 5])

$$\hat{\eta} = \left[\sum_n \frac{c_n}{n!} \, \Omega_{a_1 \cdots a_n} \Gamma^{a_1 \cdots a_n} \right] \epsilon \otimes \theta \equiv \sum_n c_n \, \Omega^{(n)} \, \epsilon \otimes \theta \tag{3.12}$$

where the constants c_n will be fixed later. Of course the Killing spinor $\hat{\eta}$ parameterize supersymmetry transformations and hence has to be globally well-defined. Assuming the same for the reduced 4- and 7-spinor implies that the differential forms $\Omega^{(n)}$ have to be globally well-defined and assuming that we have at least one Killing spinor we can construct them as in (2.4). If they are covariantly constant, equation (3.11) becomes an equation for the 7-spinor $\nabla_m^{(h)} \theta = 0$, which in turn ensures the covariantly constance of the forms Ω. In this case the space described by the metric h_{mn} has a holonomy group inside G_2. As we discussed in the previous section, the geometry of the space depends now on the number of spinors restricting more or less the holonomy. It equals G_2 if there is just one spinor fulfilling this equation, otherwise it is SU(3), SU(2) or trivial if there are two, four or eight spinors, resp. Recall, the reduction of the holonomy was related to the appearance of covariantly constant vectors yielding a factorization of the space into $\mathbb{R} \times \mathbb{X}_6$, $\mathbb{R}^3 \times \mathbb{X}_4$ or $\mathbb{R}^7$. We are interested in the case with just one Killing spinor and hence there is no covariantly constant vector and the space does not factorize. Recall, the 7-d Majorana condition for a commuting spinor θ allowed only for a 0-, 3-, 4- and a 7-form, where the 4- and 7-form are the Hodge-dual of the 3- and 0-form. Using the Γ-matrices in (2.1) we write for $\hat{\eta}$ as

$$\begin{aligned}\hat{\eta} &= \left[c_0 + c_3\Omega^{(3)} + c_4\Omega^{(4)} + c_7\Omega^{(7)}\right]\epsilon\otimes\theta \\ &= \left[(c_0 - i\,c_7\hat{\gamma}^5)\otimes\mathbb{I} + (c_3\hat{\gamma}^5 + i\,c_4)\otimes\frac{1}{3!}\varphi_{mnp}\gamma^{mnp}\right]\epsilon\otimes\theta \end{aligned} \tag{3.13}$$

where $\varphi_{mnp} = i\,\theta\gamma_{mnp}\theta$ as introduced in (2.4). Note, this expression is also true if the 7-spinor and hence the 3-form are *not covariantly constant* with respect $\nabla^{(h)}_m$, we only assumed the existence of exactly one Killing spinor θ on the 7-manifold. In fact inserting (3.12) into (3.11) one finds that $\nabla^{(h)}_m\theta$ is proportional to the covariant derivative of the differential forms. In order to explore the equations further one performs again the G_2 decomposition using the projectors $\mathbb{P}_{\mathbf{7}/\mathbf{14}}$ and finds the single Killing spinor as solution of equation (2.10). But equation (2.9), which was the projection of the spin connection onto the **7**, does not hold anymore and reflects exactly the appearance of non-trivial G-structures which are also related to covariantly non-constant differential forms. Using the projector constraint in (2.11) we find for (3.13)

$$\hat{\eta} = \Big[(c_0 - i\,c_7\hat{\gamma}^5)\otimes\mathbb{I} + i\,7\,(c_3\hat{\gamma}^5 + i\,c_4)\otimes\mathbb{I}\Big]\epsilon\otimes\theta \tag{3.14}$$

$$= \Big[(c_0 - 7c_4)(\mathbb{I}\otimes\mathbb{I}) - i\,(c_7 - 7c_3)(\hat{\gamma}^5\otimes\mathbb{I})\Big]\epsilon\otimes\theta\,. \tag{3.15}$$

With this expression, we have finally to look for solutions of (3.8) without imposing any Weyl condition on the 4-spinor ϵ and get two equations: one proportional to $\mathbb{I}\otimes\mathbb{I}$ and the other proportional to $\hat{\gamma}^5\otimes\mathbb{I}$. The first one reads

$$(c_0 - 7\,c_4)\,\gamma_m^{\ n}\partial_n A\,\theta = \frac{1}{12}\,i\,(c_7 - 7\,c_3)\,e^{3B}F_m\theta \tag{3.16}$$

and the other can be treated with the similar arguments which now follows. Contracting this equation with the Majorana spinor θ and due to the arguments after equation (2.4) we find a zero on the lhs and the rhs yields

$$(c_7 - 7\,c_3)\,F_{mnpq}\varphi^{npq} = 0\,. \tag{3.17}$$

On the other hand, if one multiplies (3.16) first with γ^l followed by the contraction with θ gives after using the relation (3.6)

$$(c_0 - 7\,c_4)\varphi^{lmn}\partial_n A = \frac{1}{12}\,i\,(c_7 - 7\,c_3)\,e^{3B}F_{mpqr}\psi^{pqrl}$$

which becomes after contraction with φ_{klm} (see eq. (2.8))

$$(c_0 - 7\,c_4)\,\partial_n A \sim (c_7 - 7\,c_3)\,F_{npqr}\varphi^{pqr}\,. \tag{3.18}$$

Thus, this equation can be solved only if $c_0 - 7\,c_4 = 0$ by supposing non-trivial fluxes and warp factor. The second equation which is proportional to $\hat{\gamma}^5\otimes\mathbb{I}$ leads to the condition $c_7 - 7\,c_3 = 0$. These conditions however mean that the spinor $\hat{\eta}$ in (3.15) is trivial. On the other hand if $\partial_n A = 0$ it follows that $F_{mnpq}\varphi^{npq} = F_{mnpq}\psi^{npqr} = 0$. Next one decomposes

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the 4-form as $\mathbf{35} = \mathbf{1}+\mathbf{7}+\mathbf{27}$ and finds that all components have to vanish identically. That for constant warp factor the fluxes have to be trivial has been found also by other authors, see e.g. [1, 5].

Thus, we come to the conclusion that it is not possible to turn on 4-form fluxes while allowing for only one 7-spinor θ.

4. Discussion

In this paper we were interested in warp compactifications of M-theory on a 7-manifold that yields a flat 4-d Minkowski space and preserve four supercharges. In the absence of 4-form fluxes this reduces the holonomy of the 7-manifold to G_2 and if there are more unbroken supercharges the holonomy group is further reduced. An obvious question is: Starting from a given G_2 manifold, can one turn on 4-form fluxes without changing the topology of the 7-manifold? Our calculation showed that this is not possible. In fact, the obstruction for 4-form fluxes was related to strong constraints coming from the fact that the 7-manifolds allows for only one Killing spinor θ yielding only 3- and 4-forms that are globally well-defined. Concretely, we considered a general decomposition of the 11-d spinor into a 4-spinor ϵ and 7-spinor θ. We assumed only one 7-spinor and showed that the 11-d Killing spinor equations can be solved only for a trivial spinor (broken supersymmetry) or trivial 4-form fluxes. This conclusion was reached under the further assumption that the 4-d external space is flat, but it is known that 4-form fluxes can be turned on if the 4-d space is anti de Sitter [20] and therefore our result can also be interpreted that a non-trivial 4-form flux on a G_2-manifold will always curve the external space or breaks supersymmetry.

As next question one may ask: What happens if there are more 7-spinors, which cannot be G_2 singlets, but are singlets under the decomposition under SU(3) or under SU(2)? With already two spinors θ^l one can build a Killing vector $V \sim i(\theta^1 \gamma_m \theta^2)$ and incorporating this vector into the ansatz (3.12) allows in fact for consistent solutions of the equations, which can be seen by repeating the calculations. But note, the existence of a Killing vector means that the 7-manifold is effectively reduced to a 6-manifold and it is known from 10-d string theory that there are warp compactifications to flat 4-d Minkowski space if one takes into account non-trivial SU(3) structures [14, 18, 16, 19]. In fact, following the procedure done in [28] it is straightforward to the see the appearance of torsion terms coming from the contraction of the 4-form with Killing vector, which in fact gives exactly the NS-3-form in string theory. E.g. having two Killing spinors one obtains in addition to one Killing vector also one 2-form, which is however not exact, but: $\partial_{[p}\Omega_{mn]} = -F_{mnpq}V^q$ [28]; similar expression exist also for the other forms. Let us also mention, that we would not see it as a G_2 compactifications, since if one turns off the fluxes the 7-manifold will have at most SU(3) holonomy, i.e. the geometry is given by $\mathbb{R} \otimes \mathbb{CY}_3$. But let us stress, this case does not mean that the solution has more supersymmetry in general! In the absence of fluxes one will of course have eight unbroken supercharges, which correspond to an $\mathcal{N}$=2, D=4 vacuum, but the presence of fluxes may result into an additional constraint on the two 4-spinors yielding an $\mathcal{N}$=1 vacuum. E.g. the solutions discussed in [9, 10] have exactly four unbroken supercharges and corresponds to a M-theory warp compactification with non-trivial 4-form fluxes. It would be interesting to work out in detail the relation of M-theory compactification with fluxes and possible SU(3) structures, see also [34].

We conclude, in M-theory (warp) compactifications to flat 4-dimensional Minkowski space, 4-form fluxes can be turned on only, if the 7-manifolds support at least two Killing spinors θ^l yielding to a reduction to a 6-manifold with non-trivial SU(3) structures.

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